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$\cos \frac{1}{4}n\pi$, $\cos \frac{3}{4}n\pi$, $\cos \frac{5}{4}n\pi$, $\cos \frac{7}{4}n\pi$, $\cos \frac{9}{4}n\pi$, $\cos \frac{11}{4}n\pi$ and $\cos \frac{13}{4}n\pi$ are the ratio.

Now $\cos \frac{1}{4}n\pi = \mp 1$, according as n is *odd* or *even*, and $\cos \frac{13}{4}n\pi = \cos \frac{1}{4}n\pi$; $\cos \frac{9}{4}n\pi = \cos \frac{5}{4}n\pi$.

Hence we have (6), $(\cos n\theta \pm 1)(64\cos^6 n\theta \mp 64\cos^5 n\theta - 48\cos^4 n\theta \pm 48\cos^3 n\theta + 8\cos^2 n\theta \mp \cos n\theta + 1) = 0$, according as n is *odd* or *even*.

$\therefore$ (7), $2(\cos \frac{1}{4}n\pi + \cos \frac{3}{4}n\pi + \cos \frac{5}{4}n\pi) = \pm \frac{8}{4} = \pm 1$. Or $\cos \frac{1}{4}n\pi + \cos \frac{3}{4}n\pi + \cos \frac{5}{4}n\pi = \pm \frac{1}{2}$, according as n is *odd* or *even*.

We might deduce a number of equally interesting results, thus,

$$(\cos \frac{1}{4}n\pi \cdot \cos \frac{3}{4}n\pi \cdot \cos \frac{5}{4}n\pi)^2 = \frac{1}{8}.$$

$\therefore \cos \frac{1}{4}n\pi \cdot \cos \frac{3}{4}n\pi \cdot \cos \frac{5}{4}n\pi = \pm \frac{1}{2}$, when n is either *odd* or *even*, etc.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University of Mississippi.

A rectangular hyperbola cannot be cut from a right circular cone if the angle at its vertex is less than a right angle.

II. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Assume axes of coördinates at right angles.

(a) The equation of the surface of a cone with axis of z as axis of cone, and origin at the vertex of cone is

$$x^2 + y^2 - z^2 \tan^2 \frac{1}{2}v = 0 \dots \dots \dots (1)$$

where v = angle at vertex.

(b) The equation of a plane to same axes and origin as above, in terms of its direction cosines and perpendicular from origin is

$$lx + my + nz = \phi \dots \dots \dots (2).$$

Eliminate z between (1) and (2), and then we have the conic

$$(n^2 - l^2 \alpha^2)x^2 - 2lmxy\alpha^2 + y^2(n^2 - m^2 \alpha^2) + 2pl\alpha^2 x + 2pm\alpha^2 y - p^2 \alpha^2 = 0 \dots (3)$$

in which α is substituted for $\tan \frac{1}{2}v$.

In order that this conic may be an equilateral hyperbola, the angle between its asymptotes

$$(n^2 - l^2 \alpha^2)x^2 - 2lm\alpha^2 xy + (n^2 - m^2 \alpha^2)y^2 = 0$$

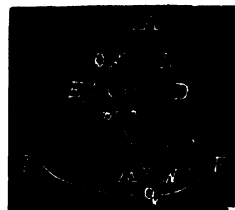
must be a right angle, the condition for which is (for rectangular axes)

$$n^2 - l^2 \alpha^2 + n^2 - m^2 \alpha^2 = 0, \text{ or } \alpha^2 = 2n^2 / l^2 + m^2 \dots \dots \dots (4).$$

Now, in order that the plane above considered shall cut out the hyperbola, the angle whose direction cosine is n must be less than $\frac{1}{2}v$; that is to say, l and m must both be less than n . Hence, $l^2 + m^2$ is necessarily less than $n^2 + n^2$ or $2n^2$; or the fraction (4) is an improper fraction, whence $\alpha^2 (= \tan^2 \frac{1}{2}v)$ is greater than unity. This establishes that $\frac{1}{2}v$ is greater than 45° , and v is greater than 90° .
Q. E. D.

III. Solution by GEORGE LILLEY, LL. D., Portland, Oregon.

Let ABF be a section of the cone made by the plane of the paper passing through its axis AM ; $OPQNH$ any section of the cone made by a plane perpendicular to the plane ABF . Pass a plane through P at right angles to AM cutting the plane ABF in DE . Draw OL parallel to BF , and HK parallel to AF . Let $\angle MAB = \alpha$, $\angle AOH = \theta$, $AO = c$, $OH = x$, and $HP = y$.



$$HP^2 = HD \times HE. \quad HE = \frac{x \sin \theta}{\cos \alpha},$$

$$DH = LK = 2c \sin \alpha - \frac{x \sin(\theta + 2\alpha)}{\cos \alpha}.$$

$$\therefore y^2 = \frac{2c \sin \theta \sin \alpha}{\cos \alpha} x - \frac{\sin \theta \sin(\theta + 2\alpha)}{\cos^2 \alpha} x^2.$$

The section represented by the equation is any hyperbola when $\theta + 2\alpha$ is greater than 180° . Comparing the equation with $y^2 = -\frac{b^2}{a}x + \frac{b^2}{a^2}x^2$, we have

$$\frac{2b^2}{a} = \frac{2c \sin \alpha \sin \theta}{\cos \alpha}, \quad \frac{b^2}{a^2} = \frac{\sin \theta \sin(\theta + 2\alpha)}{\cos^2 \alpha}.$$

$$\therefore 2a = \frac{c \sin 2\alpha}{\sin(\theta + 2\alpha)}, \quad b^2 = \frac{c^2 \sin^2 \alpha}{\sin(\theta + 2\alpha)}.$$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{1 - \sin^2(\theta + \alpha) - 2\sin^2 \alpha}{\cos^2 \alpha}, \text{ where } e \text{ is the excentricity of the}$$

$$\text{hyperbola. Or } 1 = \frac{1 - \sin^2(\theta + \alpha) - 2\sin^2 \alpha}{e^2 \cos^2 \alpha}.$$

$e^2 \cos^2 \alpha$ must not be greater than unity. But $e^2 = 2$; therefore, $\cos^2 \alpha$ must not be greater than $\frac{1}{2}$, and α must not be less than 45° . Hence, the angle at the vertex of the cone must not be less than a right angle; therefore, it is greater than a right angle.

It may, however, be equal to that angle.

Note on the angle between the asymptotes of the hyperbola.

Let ϕ = the angle between the asymptotes, and we have $\sec \frac{1}{2}\phi = e$, where e is the excentricity of the hyperbola.

$\sec^2 \frac{1}{2}\phi \cos^2 \alpha = e^2 \cos^2 \alpha$, or $\frac{\cos^2 \alpha}{\cos^2 \frac{1}{2}\phi} = e^2 \cos^2 \alpha$, but $e^2 \cos^2 \alpha$ must not be greater than unity, see solution of problem 63. Hence, $\cos \frac{1}{2}\phi$ must not be less than $\cos \alpha$ and α must not be less than $\frac{1}{2}\phi$; or the angle at the vertex of the right circular cone, from which the hyperbola is cut, must not be less than the angle between the asymptotes. *It may, however, be equal to that angle.*

IV. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

If the axis of the cone be the axis z ; h , the distance of the vertex from the origin, and θ the semi-angle at the vertex, the equation of the cone is

$$(x^2 + y^2) \tan^2(90^\circ - \theta) = (h - z)^2.$$

The section of this cone made by a plane through the axis y is a conic section, and if the angle which the plane makes with xy be ϕ and the curve of intersection be referred to axes in its own plane, its equation is

$$y^2 \tan^2(90^\circ - \theta) + x^2 \cos^2 \phi [\tan^2(90^\circ - \theta) - \tan^2 \phi] + 2hx \sin \phi - h^2 = 0.$$

If this is a rectangular hyperbola, then

$$\tan^2(90^\circ - \theta) = \cos^2 \phi [\tan^2 \phi - \tan^2(90^\circ - \theta)].$$

$$\therefore \tan \phi = \pm \frac{\sqrt{2 \sin(90^\circ - \theta)}}{\sqrt{\cos^2(90^\circ - \theta)}}. \text{ But } \phi \text{ is real.}$$

$$\therefore 90^\circ - \theta < 45^\circ. \quad \therefore -\theta < -45^\circ. \quad \therefore 2\theta > 90^\circ.$$

An hyperbola may also be cut from this cone by a plane parallel to the axis z , Its equation then is, if the cutting plane is $y = a$,

$$(x^2 + a^2)\tan^2(90^\circ - \theta) = (h - z)^2.$$

If this be a rectangular hyperbola,

$$\tan^2(90^\circ - \theta) = 1 \quad \tan(90^\circ - \theta) = 1 \quad (-1 \text{ makes } \theta \text{ negative}).$$

$$\therefore 90^\circ - \theta = 45^\circ. \quad \therefore \theta = 45^\circ. \quad \therefore 2\theta = 90^\circ.$$

This problem was also solved in an excellent manner by *G. B. M. ZERR*.

64. Proposed by *WILLIAM E. HEAL*, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C' . Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C'' . Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC .

Solution by *G. B. M. ZERR*, A. M., Ph. D., Texarkana, Arkansas-Texas.

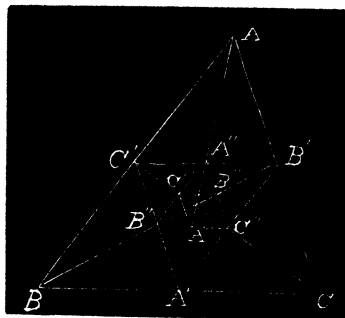
Using trilinear coordinates we have $\beta - \gamma = 0$, $\gamma - \alpha = 0$, $\alpha - \beta = 0$ for the equations to AA', BB', CC' respectively.

$$\left(0, \frac{2\Delta}{b+c}, \frac{2\Delta}{b+c}\right), \quad \left(\frac{2\Delta}{a+c}, 0, \frac{2\Delta}{a+c}\right),$$

$$\left(\frac{2\Delta}{a+b}, \frac{2\Delta}{a+b}, 0\right)$$

are the coordinates of A', B', C' respectively.

$\therefore \alpha + \beta - \gamma = 0$, $\alpha + \gamma - \beta = 0$, $\beta + \gamma - \alpha = 0$ are the equations to $A'B', A'C', B'C'$ respectively.



$$\left(\frac{4\Delta}{2a+b+c}, \frac{2\Delta}{2a+b+c}, \frac{2\Delta}{2a+b+c}\right), \left(\frac{2\Delta}{a+2b+c}, \frac{4\Delta}{a+2b+c}, \frac{2\Delta}{a+2b+c}\right),$$

$$\left(\frac{2\Delta}{a+b+2c}, \frac{2\Delta}{a+b+2c}, \frac{4\Delta}{a+b+2c}\right)$$

are the coordinates of A'', B'', C'' respectively.

$\therefore \alpha + \beta - 3\gamma = 0$, $\alpha + \gamma - 3\beta = 0$, $\beta + \gamma - 3\alpha = 0$ are the equations to $A''C'', A''B'', B''C''$ respectively.

$$\left(\frac{4\Delta}{2a+3b+3c}, \frac{6\Delta}{2a+3b+3c}, \frac{6\Delta}{2a+3b+3c}\right), \left(\frac{6\Delta}{3a+2b+3c}, \frac{4\Delta}{3a+2b+3c}, \frac{6\Delta}{3a+2b+3c}\right),$$